TAUTOLOGICAL ALGEBRAS OF MODULI SPACES : SURVEY AND PROSPECT (Topology, Geometry and Algebra of low-dimensional manifolds)

数理解析研究所講究録
第 1991 巻 2016 年 130-139

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1. INTRODUCTION

This paper is based on the author's talk at the Numazu workshop which in turn was based on a joint work with Takuya Sakasai and Masaaki Suzuki. It reproduces the actual talk rather closely. The contents are roughly as follows. In §2, we briefly recall known results about the tautological algebras of various moduli spaces. In particular, an important conjecture, called the Faber conjecture, concerning the tautological algebra of the moduli space of curves is mentioned. In §3, we summarize our former topological approach to the tautological algebra. In §4, a complete description is given how the space of symplectic invariant tensors degenerates with respect to the genus. Then in §5, we sketch our new results which are obtained by combining a theorem of Manivel on a plethysm of certain GL-representations and the result mentioned in §4. Finally in §6, we present several open problems.

2. TAUTOLOGICAL ALGEBRAS OF MODULI SPACES $G_k(\mathbb{C}^n), \mathbf{A}_g, \mathbf{M}_g$

In this section, we recall known results about the tautological algebras of various moduli spaces. First we consider the elementary case of the Grassmann manifold

$$G_k(\mathbb{C}^n) = \{V \subset \mathbb{C}^n;\ k\text{-dimensional linear subspace}\}.$$

Let $\xi \to G_k(\mathbb{C}^n)$ denote the tautological bundle over $G_k(\mathbb{C}^n)$ which is a k-dimensional vector bundle and we have the following exact sequence

$$0 \to \xi \to G_k(\mathbb{C}^n) \times \mathbb{C}^n \to Q \to 0$$

where Q denotes the quotient bundle. We denote by $c_1(\xi), \dots, c_k(\xi) \in H^*(G_k(\mathbb{C}^n); \mathbb{Z})$ the Chern classes of ξ.

Theorem 2.1 (well-known). *We have the following presentation*

$$H^*(G_k(\mathbb{C}^n); \mathbb{Q}) \cong \mathbb{Q}[c_1(\xi), \dots, c_k(\xi)]/\mathit{relations}$$

$$\mathit{relations:}\ c_i(Q) = \left[\frac{1}{1 + c_1(\xi) + \cdots + c_k(\xi)}\right]_{2i} = 0 \ \mathit{for\ all}\ i > n-k$$

and it satisfies Poincaré duality of $\dim = 2k(n-k)$.

Next, let $\mathfrak{h}_g$ denote the Siegel upper half space on which the Siegel modular group $\mathrm{Sp}(2g, \mathbb{Z})$ acts properly discontinuously. The quotient space

$$\mathbf{A}_g = \mathfrak{h}_g/\mathrm{Sp}(2g, \mathbb{Z})$$

The author was partially supported by KAKENHI (No.15H03618), Japan Society for the Promotion of Science, Japan, Received December 31, 2015.

is the moduli space of principally polarized abelian varieties. The Siegel modular group plays the role of the orbifold fundamental group of this moduli space, thus

$$\mathrm{Sp}(2g,\mathbb{Z}) \cong \pi_1^{\mathrm{orb}} \mathbf{A}_g.$$

On the other hand, $\mathrm{Sp}(2g,\mathbb{Z})$ is contained in $\mathrm{Sp}(2g,\mathbb{R})$ as a discrete subgroup and the maximal compact subgroup of the latter group is the unitary group $U(g)$. Hence we have the Chern classes

$$c_i \in H^*(\mathbf{A}_g;\mathbb{Q}) \cong H^*(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Q}).$$

Then the tautological algebra in cohomology of $\mathbf{A}_g$ is defined as

$$\mathcal{R}^*(\mathbf{A}_g) = \text{subalgebra of } H^*(\mathbf{A}_g;\mathbb{Q}) \text{ generated by } c_i\text{'s}.$$

Theorem 2.2 (van der Geer [5], true at the Chow algebra level). *The following presentation holds*

$$\mathcal{R}^*(\mathbf{A}_g) \cong \mathbb{Q}[c_1,\dots,c_g]/\mathit{relations}$$

where the relations are described as: (i) $p_i = 0$ *(Pontrjagin classes)* (ii) $c_g = 0$. *Also it satisfies Poincaré duality of* $\dim = g(g-1)$.

It is also known that, additively $\mathcal{R}^*(\mathbf{A}_g) \cong H^*(S^2 \times S^4 \times \cdots \times S^{2g-2};\mathbb{Q})$.

Finally we consider the moduli space of curves. Let $e_i \in H^{2i}(\mathcal{M}_g;\mathbb{Q})$ denote the MMM tautological class ([27][20][19]) where $\mathcal{M}_g$ denotes the mapping class group of a closed oriented surface of genus g. Then the tautological algebra of $\mathcal{M}_g$ is defined as

$$\mathcal{R}^*(\mathcal{M}_g) = \text{subalgebra of } H^*(\mathcal{M}_g;\mathbb{Q}) \text{ generated by } e_i\text{'s}.$$

Let $\mathcal{A}^i(\mathbf{M}_g)$ be the Chow algebra of the moduli space of curves of genus g defined by Mumford [27] and let

$$\kappa_i \in \mathcal{A}^i(\mathbf{M}_g)$$

be the Mumford kappa class. Then the tautological algebra of $\mathbf{M}_g$ is defined as

$$\mathcal{R}^*(\mathbf{M}_g) = \text{subalgebra of } \mathcal{A}^*(\mathbf{M}_g) \text{ generated by } \kappa_i\text{'s}.$$

There exists a canonical surjection $\mathcal{R}^*(\mathbf{M}_g) \to \mathcal{R}^*(\mathcal{M}_g)$ $(\kappa_i \mapsto (-1)^{i+1}e_i)$ and the latter group is also called the tautological algebra in cohomology of $\mathbf{M}_g$.

Conjecture 2.3 (Faber [3]). (1) *Gorenstein conjecture, including Poincaré duality*

$$\mathcal{R}^*(\mathbf{M}_g) \cong H^*(\text{“}\textit{smooth projective variety}\text{” of } \dim = g-2;\mathbb{Q})?$$

(2) $\mathcal{R}^*(\mathbf{M}_g)$ *is generated by the first* $[g/3]$ MMM-*classes with no relations in degrees* $\leq [g/3]$.

(3) *Explicit formula for the intersection numbers, namely proportionality in degree* $g-2$: $\mathcal{R}^{g-2}(\mathbf{M}_g) \cong \mathbb{Q}$ *(proved by Looijenga* [15] *and Faber* [3]*)*.

There are generalizations of this conjecture to the cases of $\overline{\mathbf{M}}_{g,n}, \mathbf{M}^{\mathrm{ct}}_{g,n}, \mathbf{M}^{\mathrm{rt}}_{g,n}$ etc. and many results due to many people, including Looijenga, Faber, Zagier, Getzler, Pandharipande, Vakil, Graber, Lee, Randal-Williams, Pixton, Liu, Xu, Yin, have been obtained (we refer to Faber's survey paper [4] for details as well as references). Returning to the above original conjecture, we have the following.

(1) is still open, Faber (and Faber and Pandharipande) verified the claim for $g \leq 23$ using the Faber-Zagier relations. On the other hand, Pandharipande and Pixton [28] showed that the Faber-Zagier relations are actual ones.

(2) was proved by M. [23] at the cohomological level, and later by Ionel [10] at the Chow algebra level. No relation part is due to Harer [9] (and improvements by Ivanov and Boldsen).

(3) There are three proofs, first by Givental [6], and the others by Liu-Xu [14] and Buryak-Shadrin [2].

More recently, Faber and Pandharipande found that some new situation happens for $g \geq 24$ (again see [4] for details).

3. Topological study of the tautological algebra of $\mathbf{M}_g$

In this section, we recall a topological approach of investigating the structure of the tautological algebra. Let Σ_g be a closed oriented surface of genus g (≥ 1) as before. We denote $H_1(\Sigma_g;\mathbb{Q})$ simply by $H_\mathbb{Q}$ which is the fundamental representation of $\mathrm{Sp} = \mathrm{Sp}(2g,\mathbb{Q})$. Let

$$\mu : H_\mathbb{Q} \otimes H_\mathbb{Q} \to \mathbb{Q}$$

be the intersection pairing. If we fix a symplectic basis of $H_\mathbb{Q}$, then as is well known there exists an isomorphism

$$\mathrm{Aut}\,(H_\mathbb{Q},\mu) \cong \mathrm{Sp}(2g,\mathbb{Q}).$$

The Torelli group is the subgroup of $\mathcal{M}_g$ defined by

$$\mathcal{I}_g = \mathrm{Ker}\,(\mathcal{M}_g \xrightarrow{\rho_0} \mathrm{Aut}\,(H_\mathbb{Q},\mu) \cong \mathrm{Sp}(2g,\mathbb{Q}))$$

where ρ_0 denotes the natural homomorphism induced by the action of $\mathcal{M}_g$ on $H_\mathbb{Q}$.

Theorem 3.1 (Johnson [11]).

$$H_1(\mathcal{I}_g;\mathbb{Q}) \cong \Lambda^3 H_\mathbb{Q}/H_\mathbb{Q} \quad (g \geq 3).$$

Let us use the following notation:

$$U_\mathbb{Q} := \Lambda^3 H_\mathbb{Q}/H_\mathbb{Q} = \text{irrep. } [1^3]_{\mathrm{Sp}}.$$

In [21], a linear representation

$$\rho_1 : \mathcal{M}_g \to H_1(\mathcal{I}_g;\mathbb{Q}) \rtimes \mathrm{Sp}(2g,\mathbb{Q})$$

of $\mathcal{M}_g$ was constructed and it induces the following homomorphism

$$\Phi : H^*(U_\mathbb{Q} = \Lambda^3 H_\mathbb{Q}/H_\mathbb{Q})^{\mathrm{Sp}} \to H^*(\mathcal{M}_g;\mathbb{Q}).$$

Theorem 3.2 (Kawazumi-M. [12]).

$$\mathrm{Im}\,\Phi = \mathcal{R}^*(\mathcal{M}_g) = \mathbb{Q}[\text{MMM-}\textit{classes}]/\textit{relations}.$$

Here recall that Madsen and Weiss [17] determined Harer's stable cohomology group ([9]) of the mapping class group to be

$$H^*(\mathcal{M}_\infty;\mathbb{Q}) = \mathbb{Q}[\text{MMM-classes}].$$

Then by analyzing the natural action of $\mathcal{M}_g$ on the *third* nilpotent quotient of $\pi_1\Sigma_g$, the author constructed in [22] the following commutative diagram

$$\begin{array}{ccc}
\pi_1\Sigma_g & \longrightarrow & [1^2]_{\mathrm{Sp}}\tilde{\times} H_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\mathcal{M}_{g,*} & \xrightarrow{\tilde{\rho}_2} & (([1^2]_{\mathrm{Sp}}^{\mathrm{torelli}} \oplus [2^2]_{\mathrm{Sp}})\tilde{\times} \wedge^3 H_{\mathbb{Q}}) \rtimes \mathrm{Sp}(2g,\mathbb{Q}) \\
\downarrow{\scriptstyle p} & & \downarrow \\
\mathcal{M}_g & \xrightarrow{\rho_2} & ([2^2]_{\mathrm{Sp}}\tilde{\times} U_{\mathbb{Q}}) \rtimes \mathrm{Sp}(2g,\mathbb{Q}).
\end{array}$$

Here $\mathcal{M}_{g,*} = \pi_0 \operatorname{Diff}_+(\Sigma_g, *)$ denotes the mapping class group of Σ_g relative to the base point $* \in \Sigma_g$ and $[2^2]_{\mathrm{Sp}} \subset H^2(U_{\mathbb{Q}})$ is the summand identified by Hain [7].

Theorem 3.3 (Kawazumi-M. [13]). *In a certain stable range, the homomorphism ρ_2^* on cohomology induces an isomorphism*

$$(H^*(U_{\mathbb{Q}})/([2^2]_{\mathrm{Sp}}))^{\mathrm{Sp}} \cong \mathbb{Q}[\text{MMM-}\mathit{classes}].$$

Similarly, in a certain stable range, $\tilde{\rho}_2^*$ induces an isomorphism

$$(H^*(\wedge^3 H_{\mathbb{Q}})/([1^2]_{\mathrm{Sp}}^{\mathit{torelli}} \oplus [2^2]_{\mathrm{Sp}}))^{\mathrm{Sp}} \cong \mathbb{Q}[e, \text{MMM-}\mathit{classes}].$$

4. Degeneration of symplectic invariant tensors

Let us consider the Sp-invariant subspace

$$(H_{\mathbb{Q}}^{\otimes 2k})^{\mathrm{Sp}}$$

of the tensor product $H_{\mathbb{Q}}^{\otimes 2k}$. We analyze the structure of this space completely. Consider the following mapping

$$\mu^{\otimes 2k} : H_{\mathbb{Q}}^{\otimes 2k} \otimes H_{\mathbb{Q}}^{\otimes 2k} \to \mathbb{Q}$$

defined by

$$(u_1 \otimes \cdots \otimes u_{2k}) \otimes (v_1 \otimes \cdots \otimes v_{2k}) \mapsto \Pi_{i=1}^{2k} \mu(u_i, v_i) \quad (u_i, v_i \in H_{\mathbb{Q}}).$$

Clearly $\mu^{\otimes 2k}$ is a symmetric bilinear form.

Theorem 4.1 (M. [25]). *The symmetric pairing $\mu^{\otimes 2k}$ on $(H_{\mathbb{Q}}^{\otimes 2k})^{\mathrm{Sp}}$ is positive definite for any g so that it defines a metric on this space. Furthermore, there exists an orthogonal direct sum decomposition*

$$(H_{\mathbb{Q}}^{\otimes 2k})^{\mathrm{Sp}} \cong \bigoplus_{|\lambda|=k,\ h(\lambda)\le g} U_\lambda$$

where for a Young diagram λ, $|\lambda|$ denotes the number of boxes and $h(\lambda)$ denotes the number of rows. Also

$$U_\lambda \cong (\lambda^\delta)_{\mathfrak{S}_{2k}} \text{ as an } \mathfrak{S}_{2k}\text{-module}$$

and there exists a bijective correspondence

$$\{\lambda; |\lambda| = k\} \overset{\text{bijective}}{\Leftrightarrow} \{\mu_\lambda; |\lambda| = k\} \quad \text{eigenvalues}.$$

Table 1 below indicates how the space $(H_{\mathbb{Q}}^{\otimes 2k})^{\mathrm{Sp}}$ degenerates according to the genus g changes from the stable range $g \geq 3k$ to $g = 3k-1, 3k-2, \ldots, 1$.

TABLE 1. Orthogonal decomposition of $(H_{\mathbb{Q}}^{\otimes 6k})^{\mathrm{Sp}}$

λ	μ_λ (eigenvalue of U_λ)	g for $U_\lambda = \{0\}$
$[3k]'$	$(2g-6k+2)\cdots(2g-2)2g$	$g \leq 3k-1$
$[3k-1,1]'$	$(2g-6k+4)\cdots(2g-2)2g(2g+1)$	$g \leq 3k-2$
$[3k-2,2]'$	$(2g-6k+6)\cdots 2g(2g-1)(2g+1)$	$g \leq 3k-3$
$[3k-2,1^2]'$	$(2g-6k+6)\cdots 2g(2g+1)(2g+2)$	$g \leq 3k-3$
$[3k-3,3]'$	$(2g-6k+8)\cdots 2g\cdots(2g-3)$	$g \leq 3k-4$
$\cdots$		

Remark 4.2. Related eigenvalues already appeared in Hanlon-Wales [8] in the context of Brauer's centralizer algebras.

Now we consider the following mappings

$$(H_{\mathbb{Q}}^{\otimes 6k})^{\mathrm{Sp}} \xrightarrow{\text{surj.}} (\wedge^{2k} U_{\mathbb{Q}})^{\mathrm{Sp}} \xrightarrow{\text{surj.}} \mathcal{R}^{2k}(\mathcal{M}_g)$$

together with the following degenerations of symplec invariant tensors

$$\begin{aligned}
&[3k]' \longmapsto 0 \quad (g \leq 3k-1) \quad (\textbf{enough} \text{ to prove Faber conjecture (2)})\\
&[3k-1,1]' \longmapsto 0 \quad (g \leq 3k-2)\\
&[3k-2,2]'[3k-2,1^2]' \longmapsto 0 \quad (g \leq 3k-3)\\
&[3k-3,3]'[3k-3,21]'[3k-3,1^3]' \longmapsto 0 \quad (g \leq 3k-4)\\
&[6k-8,8]'[6k-8,62]'\cdots[6k-8,2^4]' \longmapsto 0 \quad (g \leq 3k-5).
\end{aligned}$$

In this way, we obtain many (hopefully all? the) relations and we proposed the following.

Conjecture 4.3 (M. [24]).

$$\begin{aligned}
\mathcal{R}^*(\mathcal{M}_g) &\cong (\wedge^* U_{\mathbb{Q}}/([2^2]_{\mathrm{Sp}}))^{\mathrm{Sp}},\\
\mathcal{R}^*(\mathcal{M}_{g,*}) &\cong \left(\wedge^*(\wedge^3 H_{\mathbb{Q}})/([1^2]_{\mathrm{Sp}}^{torelli} \oplus [2^2]_{\mathrm{Sp}})\right)^{\mathrm{Sp}}.
\end{aligned}$$

5. PLETHYSM OF GL REPRESENTATIONS AND TAUTOLOGICAL ALGEBRA

In this section, we consider certain plethysm of GL-representations. Recall that plethysm is a composition of two Schur functors. Determination of a given plethysm is a very important but extremely difficult problem and a complete answer is known for only the following four cases.

Theorem 5.1 (Formula of Littlewood). *There exists a complete description of the following plethysms*

$$\begin{aligned}
&S^*(S^2 H_{\mathbb{Q}}), \quad \wedge^*(S^2 H_{\mathbb{Q}}),\\
&S^*(\wedge^2 H_{\mathbb{Q}}), \quad \wedge^*(\wedge^2 H_{\mathbb{Q}}).
\end{aligned}$$

The following result of Manivel plays a key role in our work. Here we describe his result in a simplified form and we refer to his original paper for details.

Theorem 5.2 (Manivel [18]). *The plethysm $S^k(S^l H_{\mathbb{Q}})$ "super stabilizes" as $k \to \infty$. Furthermore the super stable decomposition of $S^\infty(S^3 H_{\mathbb{Q}})$ is given by*

$$S^*(S^2 H_{\mathbb{Q}} \oplus S^3 H_{\mathbb{Q}}).$$

We apply the well-known involution on the space of symmetric polynomials (see [16]) to the following particular case

$$H_k H_3 \overset{\text{dual}}{\Leftrightarrow} E_k E_3$$

where E_k denotes the k-th elementary symmetric polynomial and H_k denotes the k-th complete symmetric polynomial. We obtain the following result.

Theorem 5.3 (Sakasai-Suzuki-M. [26]). *Let $\wedge^k(\wedge^3 H_{\mathbb{Q}})$ be the k-th exterior power of the third exterior power of $H_{\mathbb{Q}}$ and let*

$$\wedge^k(\wedge^3 H_{\mathbb{Q}}) = \bigoplus_{\lambda, |\lambda|=3k} m_\lambda \lambda_{\mathrm{GL}}$$

be the stable irreducible decomposition as a GL*-module. Then, for any k, the mapping*

$$\wedge^k(\wedge^3 H_{\mathbb{Q}}) \longrightarrow \wedge^{k+1}(\wedge^3 H_{\mathbb{Q}})$$

induced by the operation $\lambda \mapsto \lambda^+ = [\lambda 1^3]$ is injective and bijective for the part λ^+_{GL} with $2k \leq h(\lambda) \leq 3k$. In other words, we have the inequality

$$m_\lambda \begin{cases} \leq m_{\lambda^+} & \\ = m_{\lambda^+} & (2k \leq h(\lambda) \leq 3k). \end{cases}$$

Theorem 5.4 (Sakasai-Suzuki-M. [26]). *We have determined the super stable irreducible decomposition of $\wedge^\infty[1^3]_{\mathrm{GL}}$ up to codimension* 30.

Table 2 below indicates the super stable irreducible decomposition of $\wedge^\infty[1^3]_{\mathrm{GL}}$ up to codimension 10.

Corollary 5.5 (Sakasai-Suzuki-M. [26]). *We have determined the super stable* Sp*-invariant part $(\wedge^\infty[1^3]_{\mathrm{GL}})^{\mathrm{Sp}}$ up to codimension* 30.

Table 3 below indicates the super stable Sp-invariant part $(\wedge^\infty[1^3]_{\mathrm{GL}})^{\mathrm{Sp}}$ up to codimension 10.

TABLE 2. Super stable irreducible decomposition of $\wedge^\infty[1^3]_{\mathrm{GL}}$

cod.	irreducible decomposition
0	$[1^*]$
1	$[21^*]$
2	$[2^21^*]$
3	$[2^31^*]$
4	$[2^41^*][3^21^*]$
5	$[2^51^*][32^31^*][3^221^*]$
6	$2[2^61^*]2[3^22^21^*][4^21^*]$
7	$[2^71^*][32^51^*]2[3^22^31^*][3^321^*][432^21^*][4^221^*]$
8	$2[2^81^*][32^61^*]3[3^22^41^*][3^32^21^*]2[3^41^*][432^31^*]$ $[43^221^*]2[4^22^21^*][5^21^*]$
9	$2[2^91^*][32^71^*]3[3^22^51^*]3[3^32^31^*][3^421^*][432^41^*]$ $[43^22^21^*]2[43^31^*]3[4^22^31^*][4^2321^*][542^21^*][5^221^*]$
10	$2[2^{10}1^*][32^81^*]4[3^22^61^*]2[3^32^41^*]4[3^42^21^*]2[432^51^*]2[43^22^31^*]2[43^321^*]$ $4[4^22^41^*]2[4^232^21^*]3[4^23^21^*][53^22^21^*][542^31^*][54321^*]2[5^22^21^*][6^21^*]$

TABLE 3. Super stable irred. summands of $\wedge^\infty[1^3]_{\mathrm{GL}}$ with double floors

cod.	irreducible decomposition
0	$[1^*]$
2	$[2^21^*]$
4	$[2^41^*][3^21^*]$
6	$2[2^61^*]2[3^22^21^*][4^21^*]$
8	$2[2^81^*]3[3^22^41^*]2[3^41^*]2[4^22^21^*][5^21^*]$
10	$2[2^{10}1^*]4[3^22^61^*]4[3^42^21^*]4[4^22^41^*]3[4^23^21^*]2[5^22^21^*][6^21^*]$

Let us consider the following series of mappings (see [4] for the definition of the Gorenstein quotients).

$$\mathcal{R}^*(\mathbf{M}_g) \to \mathcal{R}^*(\mathcal{M}_g) \to G^*(\mathbf{M}_g) \text{ (Gorenstein quotient)},$$
$$\mathcal{R}^*(\mathbf{M}_g^1) \to \mathcal{R}^*(\mathcal{M}_{g,*}) \to G^*(\mathbf{M}_g^1) \text{ (Gorenstein quotient)}.$$

Here $\mathbf{M}_g^1$ denotes the moduli space of curves of genus g with one marked point.

Expectation 5.6 (Faber-Zagier [3][4]; Faber-Bergvall [1], Yin [29]). *The number*

$$p(k) - \dim G^{2k}(\mathbf{M}_g) = \textit{number of relations of codimension } k$$

depends only on $\ell = 3k-1-g$ in the range $2k \leq g-2$ (i.e. $k \geq \ell+3$). Similarly the number

$$1 + p(1) + \cdots + p(k) - \dim G^{2k}(\mathbf{M}_g^1)$$

depends only on $\ell = 3k-1-g$ in the range $2k \leq g-2$ (i.e. $k \geq \ell+3$), in case $2k = g-1$, something happens?

Expectation 5.7 (continued, Faber-Zagier [3][4]; Faber-Bergvall [1], Yin [29]). *If the former part of the previous Expectation holds, then the above number can be described as*

$$a(\ell) \stackrel{?}{=} \text{number of partitions of } \ell \text{ with parts:}$$
$$1, 2, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, \ldots$$
$$n \neq 2 \text{ is excluded if } n \equiv 2 \bmod 3.$$

Similarly, if the latter part of the previous Expectation holds, then the above number can be described as

$$b(\ell) \stackrel{?}{=} \sum_{i=0,\, i\neq 3m+2}^{\ell} a(\ell - i) = a(\ell) + a(\ell-1) + a(\ell-3) + a(\ell-4) + \cdots .$$

We have the following theorems which may serve as a supporting evidences for the above expectations.

Theorem 5.8 (Sakasai-Suzuki-M. [26]). (1) *The number*

$$\tilde{a}(\ell) := p(k) - \dim\left(\wedge^{2k} U_{\mathbb{Q}} / ([2^2]_{\mathrm{Sp}})\right)^{\mathrm{Sp}}$$

depends only on $\ell = 3k-1-g$ in the range $2k \leq g-2$ (i.e. $k \geq \ell + 3$).

(2) *The number*

$$\tilde{b}(\ell) := 1 + p(1) + \cdots + p(k) - \dim\left(\wedge^{2k}(\wedge^3 H_{\mathbb{Q}}) / ([1^2]_{\mathrm{Sp}}^{torelli} \oplus [2^2]_{\mathrm{Sp}})\right)^{\mathrm{Sp}}$$

depends only on $\ell = 3k-1-g$ in the same range.

Furthermore, we have the following more precise result.

$$\text{orthogonal complement of } \left(\wedge^{2k}(\wedge^3 H_{\mathbb{Q}})\right)^{\mathrm{Sp}} \text{ in } \left(\wedge^{2k}(\wedge^3 H_{\mathbb{Q}}^{\infty})\right)^{\mathrm{Sp}} \bmod ([1^2]_{\mathrm{Sp}}^{\mathrm{torelli}} \oplus [2^2]_{\mathrm{Sp}})^{\mathrm{Sp}}$$
$$\Rightarrow \text{tautological relations in } \mathcal{R}^{2k}(\mathcal{M}_{g,*})$$
$$\text{orthogonal complement of } \left(\wedge^{2k} U_{\mathbb{Q}}\right)^{\mathrm{Sp}} \text{ in } \left(\wedge^{2k} U_{\mathbb{Q}}^{\infty}\right)^{\mathrm{Sp}} \bmod ([2^2]_{\mathrm{Sp}})^{\mathrm{Sp}}$$
$$\Rightarrow \text{tautological relations in } \mathcal{R}^{2k}(\mathcal{M}_g).$$

Theorem 5.9 (Sakasai-Suzuki-M.). *If we fix $\ell = 3k-1-g$, then all the above orthogonal complements are canonically isomorphic to each other in the range $2k \leq g-2$ (i.e. $k \geq \ell+3$).*

6. Problems

Problem 6.1. *Construct the "fundamental cycles"*

$$\mu_{g,*} \in \left(\wedge^{2g-2}(\wedge^3 H_{\mathbb{Q}}^{\langle g\rangle})\right)^{\mathrm{Sp}},$$
$$\mu_g \in \left(\wedge^{2g-4} U_{\mathbb{Q}}^{\langle g\rangle}\right)^{\mathrm{Sp}}$$

and give a topological proof of the intersection number formula.

Problem 6.2. *Give a topological proof of the Faber-Zagier relations.*

Problem 6.3. *Study the relation between our tautological relations with those of Faber-Zagier as well as those of Yin.*

Problem 6.4 (suggested by Faber [4]). *Which part (and/or in which degrees) of the following homomorphisms is isomorphic or non-isomorphic?:*

$$\mathcal{R}^*(\mathbf{M}_g) \to \mathcal{R}^*(\mathcal{M}_g) \to G^*(\mathbf{M}_g) \ (\textit{Gorenstein quotient}),$$
$$\mathcal{R}^*(\mathbf{M}_g^1) \to \mathcal{R}^*(\mathcal{M}_{g,*}) \to G^*(\mathbf{M}_g^1) \ (\textit{Gorenstein quotient}).$$

Problem 6.5 (suggested by Faber [4]). (1) *Are Faber-Zagier relations linearly independent?*

(2) *Are Faber-Zagier relations complete up to the half dimension?*

(3) *Are there more relations (above the half dimension)?*

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